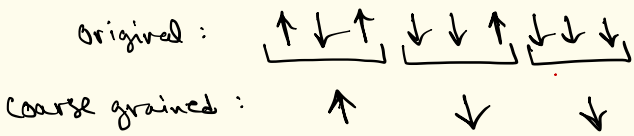
Introduction

Emergence of self-similarity in nature

Consider a magnet. Microscopically, it consists of 'spins' on a lattice: ↑ ↓ ↑ ...etc. For simplicity, let's assume that spins can only point up or down ('Ising' spins), and further assume that underlying Hamiltonian has the spin-flip symm. $S \to -S$, and the dimension $d > 1$. As discussed extensively in 210A, such magnets exhibit a 'phase transition' as a fn of temperature T: for $T > T_c$, one is in a paramagnet (i.e. disordered) phase, while for $T < T_c$, one is in a ferromagnetic (i.e. ordered) phase. Lets take snapshots of the system for three different cases: (i) $T > T_c$ (ii) $T < T_c$ (iii) $T = T_c$. An interesting processing of these images is to perform a 'coarse-graining' procedure, e.g. given a configuration of spins (this is a 1d example)

↑ ↓ ↑ ↓ ↓ ↑ ↓ ↓ ↓ , lets group spins into sets of three and apply a majority rule to obtain a coarse-grained config.



Having applied such a coarse-graining procedure repeatedly on an infinitely large system (after generalizing it to $d > 1$), one finds :

(i) $T < T_c$: repeated coarse-graining leads to an image that is all up or all down, depending on the direction of ordering.

(ii) $T > T_c$: repeated coarse-graining leads to an image that corresponds to uncorrelated sites with equal prob. of being up or down on each site.

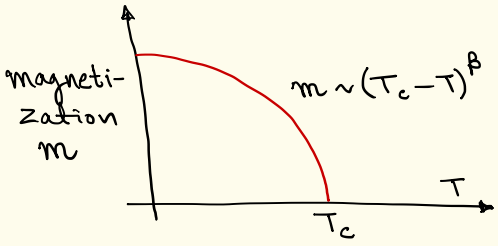
(iii) $T = T_c$: repeated coarse-graining leads to scale-invariant fractal like structures

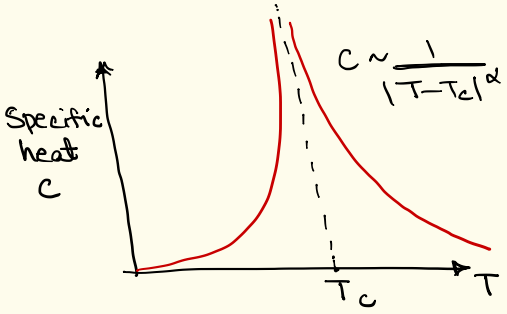
The observations (i) and (ii) are quiet intuitive and also follow from a mean-field picture. (iii) is a rather non-trivial and the bulk of the course is about understading such behaviours and its consequences.

Scale invariance and Power laws

Self-similarity / scale invariance is closely tied with the emergence of power-laws in various quantities.

e.g. in $d > 1$ Ising model:





See the ref. Nature 378, 597 (1995) posted in Canvas for some beautiful data.

Scale invariance and Universality

Perhaps the most pertinent point is that a given scale-invariant behavior corresponds to an infinitely many different microscopic models. For example, the critical behavior of all liquids at the end-point of liquid-gas phase boundary is identical. Similarly, all materials with Ising symmetry but different microscopic Hamiltonian all have same critical exponents for their order-disorder transition.

Fractals

The fact that critical points have scale invariance prompts us to study self-similar objects ('fractals'), which are interesting in their own rights. We will closely follow the chapter 1 of Cressick's book (see also McGreery's lecture notes).

Fractal dimension:

There are several different notions of fractal dim. One useful to us would be: cover the object of interest with d-balls of diameter a, and let the minimum number of balls be $N(a)$. Clearly, as $a \to 0$, we expect that $N(a) \to \infty$. The fractal dim D captures this divergence:

$$N(a) \sim \frac{1}{a^D} \quad \text{as} \quad a \to 0.$$

Examples:

Cantor set: $D = \log(2)/\log(8)$.

Sierpinski carpet: $D = \log(8)/\log(3)$.

One way to calculate the fractal dim. B:

If after one step of iteration, there are 'x' times as many objects as before that iteration, which are '$\frac{1}{y}$' times as large, then the fractal dim is $\frac{\log(x)}{\log(y)}$. Reasoning:

$$N[a] = x\, N[ya]$$

Assuming $N[a] \sim \frac{1}{a^D}$

$$\Rightarrow \quad 1 = \frac{x}{y^D} \quad \Rightarrow \quad D = \frac{\log(x)}{\log(y)}$$

Cantor set: $x = 2$, $y = 3$.

Sierpinski square carpet: $x = 8$, $y = 3$.

Sierpinski triangle carpet: $x = 3$, $y = 2$.

Random Walks and scale Invariance.

Statistically independent steps of length a_0 in arbitrary direction. After M steps,

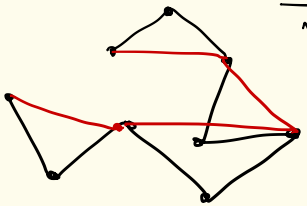
$$\langle R^2 \rangle = \left\langle \left(\sum_{i=1}^{M} \vec{r}_i \right)^2 \right\rangle = M a_0^2$$

Since $\langle \vec{r}_i^{\,2} \rangle = a_0^2$ and $\langle \vec{r}_i \cdot \vec{r}_j \rangle = 0$ when $i \neq j$.

$\Rightarrow$ root mean square displacement $\sim \sqrt{M}$ (RMS)

Fractal dimension:

Consider $M \gg 1$, and 'coarse grain' the random walk — define a new walk where each step of the new walk corresponds to n steps of the original walk. The RMS distance moved in each subwalk $= \sqrt{n}\, a_0$, and we have $\frac{M}{n}$ such subwalks.



$n = 2$

# of balls of radius a_0 needed to cover the walk, $N(a_0) = M$.

If one instead uses balls of radius $\sqrt{n}\, a_0$, then one only needs M/n balls. i.e.

$$N(\sqrt{n}\, a_0) = \frac{M}{n} = N(a_0)/n$$

$$\Rightarrow \quad N(a) = n\, N(\sqrt{n}\, a)$$

Assuming $N(a) \sim \frac{1}{a^x}$

$$\Rightarrow \quad \frac{1}{a^x} = \frac{n}{n^{x/2}\, a^x} \quad \Rightarrow \quad x = 2$$

Thus, the fractal dimension of a random walk is $D = 2$, irrespective of the space dimension in which the walk is embedded.

In this sense, random walks are scale-invariant.

> Random walk begets random walk
> $\Leftrightarrow$ Random walk as a fixed point of coarse graining + scale transformation.

Above we saw that one can coarse-grain a random walk of elementary step a_0 into a new random walk of elementary step $\sqrt{n}\, a_0$. If one zooms out (i.e. rescale) all length scales by a factor $\sqrt{n}$, then one recovers the original walk. Lets put such coarse-graining + scale transformation on a slightly more general footing.

Consider a random walk where the length of each step is picked from a Gaussian p.d.f:

$$p(\vec{r}) = \frac{1}{(2\pi\sigma_0^2)^{d/2}} e^{-\frac{r^2}{\sigma_0^2}}$$

Step#1 : Coarse-Graining

To coarse-grain this walk in the same sense as the preceding discussion (where the length of each step was fixed), let's define $\vec{r}' = \sum_{i=1}^{n} \vec{r}_i$, so that

$$p(\vec{r}') = \int dr_1\, dr_2 \cdots dr_n\; \delta(r' - \sum_{i=1}^{n} r_i)\; p(r_1)\, p(r_2) \cdots p(r_n)$$

Performing the integral, one obtains

$$p(r') = \frac{1}{[2\pi\sigma_0^2 n]^{d/2}} e^{-\frac{r'^2}{n\sigma_0^2}}$$

Therefore $p(\vec{r}')$ is almost identical to $p(\vec{r})$, except that its variance is $n\sigma_0^2$ instead of σ_0^2.

Step # 2 : Rescaling

Having performed the coarse-graining, let's rescale all lengths so that the variance of the rescaled variable is again σ_0. Note that this specific rescaling is a <u>choice</u> we are making and as it will become clearer later, is tantamount to the assumption that we are in the vicinity of the random walk with the Gaussian p.d.f. Therefore, let's define $\vec{r}\,' = \sqrt{n}\,\vec{r}\,''$

$$\Rightarrow \quad p(\vec{r}\,'') = \frac{1}{\left[2\pi\sigma_0^2\right]^{d/2}} \, e^{-\frac{r''^2}{2\sigma_0^2}}$$

Therefore $$\boxed{p(\vec{r}\,'') = p(\vec{r}\,')}$$

i.e. the system is invariant under the coarse-graining + rescaling ("Renormalization") procedure. This is equivalent to the statement that an unbiased random walk is scale-invariant, as we saw earlier.

What if we perturb the original distribution $p(\vec{r})$ a little bit while keeping it still a f^n of $|\vec{r}|$? We will now show that the renormalized (i.e. coarse grained + rescaled) pdf approaches the Gaussian p.d.f. as we iterate the renormalization procedure. This is nothing but the 'central limit theorem' which states that given a set of N variables $\{x_i\}$ with p.d.f.s $\{p(x_i)\}$, the p.d.f. of the average, $\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}$, approaches a Gaussian as $N \to \infty$.

Let's prove this using our RG procedure.

Let $\vec{r}_1, \ldots, \vec{r}_n$ have p.d.fs $p(\vec{r}_1), \ldots, p(\vec{r}_n)$, and consider the p.d.f. of $\vec{r}_1 = \sum_{i=1}^{n} \vec{r}_i$. Note that we are not considering the p.d.f. of average of $\{\vec{r}_i\}$ (i.e. we are not dividing by n at the outset), because we want to separate coarse-graining and rescaling. We assume that all $p(\vec{r}_i)$ have variance σ_0^2 and they all only depend on $|\vec{r}_i|$.

$$\Rightarrow \quad b(\vec{r}') = \int d^d r_1 \ldots d^d r_n \; \delta\left(r' - \sum_{i=1}^{n} r_i\right) b(\vec{r}_1) \cdots b(\vec{r}_n)$$

$$= \frac{1}{(2\pi)^d} \int d^d k \int d^d r_1 \cdots d^d r_n \; e^{-i\vec{k}\cdot\left(\vec{r}' - \sum_i \vec{r}_i\right)} b(\vec{r}_1) \cdots b(\vec{r}_n)$$

$$= \frac{1}{(2\pi)^d} \int d^d k \; e^{-i\vec{k}\cdot\vec{r}'} \prod_i \left\langle e^{i\vec{k}\cdot\vec{r}_i} \right\rangle$$

One may now use cumulant expansion.

$$\left\langle e^{i\vec{k}\cdot\vec{r}} \right\rangle = e^{i\left\{\langle \vec{k}\cdot\vec{r}\rangle - \frac{1}{2} k^2 \left[\langle \vec{r}^2 \rangle - \langle \vec{r} \rangle^2\right] + \cdots\right\}}$$

$$= e^{-\frac{k^2 \sigma_0^2}{2} + \cdots}$$

where ... denotes higher cumulants, e.g. the fourth-order term is proportional to $k^4 \left[\langle (\vec{r})^4 \rangle - 3 \langle \vec{r}^2 \rangle^2\right]$. Note that were $p(r_i)$ a Gaussian, this terms would vanish identically.

Combining everything, one obtains,

$$p(\vec{r}\,') = \int \frac{d^d k}{(2\pi)^d} \; e^{-i\vec{k}\cdot\vec{r}\,' - \frac{n}{2}\sigma_0^2 k^2 + \dots}$$

where (recall) $\vec{r}\,' = \sum_{i=1}^{n} r_i$ is our coarse-grained variable.

Now comes the punchline. The higher-order terms (denoted as ... above) contribute negligibly when $n \gg 1$. To see this, note that the Gaussian part dominates when $|k| \sim \frac{1}{\sqrt{n\sigma_0^2}}$. If so, let's consider the m'th order cumulent which is proportional to $k^m n$, and if the Gaussian part dominates, is therefore of the order

$$\left[\frac{1}{n\sigma_0^2}\right]^{\frac{m}{2}} n \sim n^{1-\frac{m}{2}}$$ which is

negligible for $m > 2$ and n large.

If we neglect these higher order terms, one obtains a Gaussian p.d.f.

$$p(\vec{r}\,') \sim e^{-|\vec{r}\,'|^2/2n\sigma_0^2}$$ which

after rescaling $\vec{r}\,'' = \frac{\vec{r}\,'}{\sqrt{n}}$, is a Gaussian with variance σ_0^2.

for practical reasons, RG is typically implemented in an iterative fashion, i.e. one coarse-grains a little bit, and then repeats. Therefore, let's consider a procedure where n is not large at the outset. for example, one could start with $n = 2$, coarse grain, re-scale and obtain a new pdf where the non-Gaussian terms are suppressed only a little bit.

As one iterates this procedure k times, the effective $n = 2^k$ at the k'th RG step and therefore one approaches the Gaussian p.d.f. rapidly. We say that the Gaussian p.d.f. is the <u>fixed-point</u> of R.G. Let's formulate this as an equation.

$$\sigma_0(m) = \sigma_0(1)$$

where m denotes the m'th iteration of RG. Although m is nominally an integer, one can formally consider $m = 1 + \delta l$, so that the above eqn becomes $\frac{d\sigma_0}{dl} = 0$. Such an equation is called 'RG flow' and the right hand side is called the 'β-fn' of variable σ_0. Here the β-fn is just zero, and we call such a variable 'marginal' since it doesn't flow under RG.

Lets consider a slightly less trivial example of the RG flow of non-Gaussian terms in the above p.d.f. That is given a p.d.f. for each step of a random walk, $p(r)$, whose fourier transform $p(k) = e^{-\frac{k^2\sigma^2}{2} + k^4\frac{\sigma_4}{4!} - \frac{k^6\sigma_6}{6!} + \ldots}$

Lets study the RG flow of σ_4, σ_6 etc. under coarse-graining. Note that σ_i's are just the cumulants of the p.d.f. and $\sigma_2 \equiv \sigma^2$.

As hinted above, at each step of RG, we will perform a coarse-graining by a factor of 2, and then rescale. The pdf of the coarse grained walk with variable $\vec{r}' = \vec{r}_1 + \vec{r}_2$ is

$$p(\vec{r}') = \int_{r_1, r_2} \delta(r' - (r_1 + r_2))\, p(r_1)\, p(r_2)$$

$$= \int_{k, r_1, r_2} p(r_1)\, p(r_2)\, e^{-ik\cdot[\vec{r}' - (\vec{r}_1 + \vec{r}_2)]}$$

$$= \int_k p_1(\vec{k})\, p_2(\vec{k})\, e^{i\vec{k}\cdot\vec{r}'}$$

Fourier transforming

$p(\vec{k}) = p_1(\vec{k})\, p_2(\vec{k})$. Thus, the coarse-graining looks simpler in momentum space.

Using $p_1(\vec{k}) = p_2(\vec{k}) = e^{-\frac{k^2\sigma^2}{2} + \frac{k^4\sigma_4}{4!} + \cdots}$

$$\Rightarrow p(\vec{k}) = e^{-k^2\sigma^2 + \frac{k^4\sigma_4 \times 2}{4!} + \cdots}$$

Next, we need to <u>rescale</u>. As noted above, here we have to make a <u>choice</u>. If we rescale so that the coefficient of σ^2 is unchanged compared to p_1, p_2, we are

implicitly making the assumption that we are in the vicinity of the Gaussian dominated fixed point of the RG, i.e., the non-Gaussian terms in p_1, p_2 are small.

Lets make this assumption and see where does it lead. Rescale by introducing k'' so that $k^2\sigma^2 = \frac{(k'')^2\sigma^2}{2}$ i.e. $k = \frac{k''}{\sqrt{2}}$.

In terms of k'',

$$p(k'') = e^{-\frac{k''^2\sigma^2}{2}} + \frac{(k'')^4}{(\sqrt{2})^4}\frac{\sigma_4}{4!} \times 2 + \cdots$$

By design, $\sigma_2 \equiv \sigma^2$ remains unchanged. However, σ_4 has changed.

$$\text{new } \sigma_4 = \text{old } \sigma_4 \times \frac{2}{(\sqrt{2})^4} = \frac{\text{old } \sigma_4}{2}$$

i.e. the non-Gaussian part of pdf has diminished after one step of RG, as anticipated. Repeating the RG step k times, and denoting σ_4 at step k by $\sigma_4(k)$,

$$\sigma_4(m) = \frac{\sigma_4(1)}{2^{m-1}}$$

Again, we write $m = 1 + d\ell$, so that

$$\sigma_4(1+d\ell) = \frac{\sigma_4(1)}{2^{d\ell}} \approx \sigma_4(1)\,[1 - d\ell\, \log(2)]$$

$$\Rightarrow \boxed{\frac{d\sigma_4}{d\ell} = -\sigma_4 \log(2)}$$

Thus, the β-f^n of σ_4 is $-\sigma_4 \log(2)$, and σ_4 decays to zero along the RG flow. We say σ_4 is <u>irrelevant</u> at the Gaussian fixed point. It's crucial to note that we are perturbing around the Gaussian fixed point as was determined by our choice above when we preserved the coefficient of σ_2 after rescaling. The above analysis shows that the Gaussian fixed point is <u>stable</u>.

What happens if we instead consider RG flow in the vicinity of a <u>non-Gaussian</u> fixed point? Lets again consider the equation for pdf after coarse graining,

$$p(\vec{k}) = e^{-\sigma^2 k^2 + \frac{k^4 \sigma_4 \times 2}{4!} + \dots}$$

Now consider a rescaling so that fourth order cumulant is unchanged:

$$\frac{(k'')^4 \sigma_4}{4!} = \frac{k^4 \sigma_4}{4!} \times 2$$

$$\Rightarrow \quad k'' = k\, 2^{1/4}$$

$$\Rightarrow \quad p(\vec{k}'') = e^{-\frac{\sigma^2 (k'')^2}{2} \times \frac{2}{2^{1/2}} + \frac{(k'')^4 \sigma_4}{4!} + \dots}$$

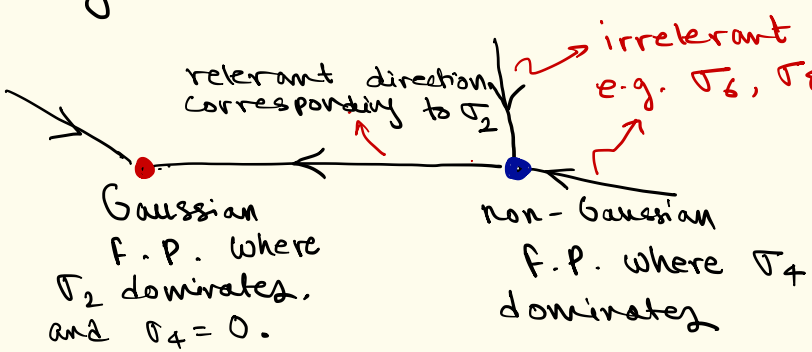
$$\Rightarrow \quad \sigma_2(m) = \sigma_2(1) \left[\sqrt{2}\right]^{m-1}$$

after m steps of RG, i.e. σ_2 grows under RG at the non-Gaussian fixed point where fourth order cumulant dominates. One may easily check that σ_6 and higher cumulants decrease towards zero under RG.

Writing it as a differential eqn.

$$\frac{d\sigma_2}{dl} = \sigma_2 \log(\sqrt{2})$$

One can depict the above analysis pictorially as,



Therefore, reaching the σ_4 non-Gaussian f.p. requires fine-tuning one variable, namely σ_2 to zero, while reaching the Gaussian f.p. does not require any fine-tuning.

One may also draw the RG flow as: